Sl. No. **C–JTT–M–TUB**

STATISTICS—II

Time Allowed : Three Hours	Maximum Marks : 200

INSTRUCTIONS

Candidates should attempt FIVE questions in ALL including Question Nos. 1 and 5, which are compulsory. The remaining THREE questions should be answered by choosing at least ONE question each from Section—A and Section—B.

The number of marks carried by each question is indicated against each.

Answers must be written only in ENGLISH.

(Symbols and abbreviations are as usual, unless otherwise indicated.)

Any essential data assumed by candidates for answering questions must be clearly stated.

All parts and sub-parts of a question being attempted must be completed before moving on to the next question.

Section—A

1. *(a)* If $\underline{l}_1'\hat{\underline{\beta}}$ and $\underline{l}_2'\hat{\underline{\beta}}$ are least squares estimators of two estimable functions $\underline{l}_1'\underline{\beta}$ and $\underline{l}_2'\underline{\beta}$ respectively, then show that—

(i) $V(\underline{l}_1'\hat{\underline{\beta}}) = \sigma^2 \underline{l}_1' \bar{S} \underline{l}_1$

(ii) $\text{cov}(\underline{l}_1'\hat{\underline{\beta}},\ \underline{l}_2'\hat{\underline{\beta}}) = \sigma^2 \underline{l}_1' \bar{S} \underline{l}_2$

where $\bar{S}$ is the generalised inverse of $S = X'X$. 8

(b) Consider the following balanced one-way classification random effects model :

$$Y_{ij} = \mu + a_i + e_{ij},\ i = 1, 2, \cdots, p,\ j = 1, 2, \cdots, n$$

where a_i's and e_{ij}'s are independent; and $a_i \sim N(0, \sigma_a^2)$ and $e_{ij} \sim N(0, \sigma_e^2)$. How would you test $H_0 : (\sigma_a^2 = 0)$ against $H_1 : (\sigma_a^2 > 0)$? 8

(c) Explain the purpose of the Gauss-Markoff theorem, and prove it. 8

(d) Let $X_1, X_2, \cdots, X_n$ be a random sample from the uniform distribution

$$f(x, \theta) = \begin{cases} \frac{1}{\theta}, & 0 \le x \le \theta \\ 0, & \text{otherwise} \end{cases}$$

Examine the consistency of the estimators $T_1 = X_{(n)}$, $T_2 = X_{(1)} + X_{(n)}$ and $T_3 = 2\overline{X}$ for estimating θ. 8

(e) Give an example of sufficient statistic which is not complete. Define bounded completeness. 8

2. (a) Consider the function of observations

$$y = C'Y,\ y_1 = C_1'Y,\ y_2 = C_2'Y,\ \cdots,\ y_k = C_k'Y$$

Show that the necessary and sufficient condition that the function y is dependant on $y_1, y_2, \cdots, y_k$ is that the coefficient vector of y is dependant on the coefficient vectors of $y_1, y_2, \cdots, y_k$. 10

(b) Under what condition the Cramer-Rao inequality reduces to the equality? Illustrate through an example how it helps in obtaining a UMVU estimate. 10

(c) Let T_0 be an MVUE, while T_1 is any other unbiased estimator with efficiency e_θ. If r_θ be the correlation coefficient between T_0 and T_1, then show that

$$r_\theta = \sqrt{e_\theta}$$ 10

(d) Explain the tests for regression coefficients in short. 10

3. (a) Let X_1, X_2, X_3 are three uncorrelated random variables having common variance σ^2. If

$$E(X_1) = \theta_1 + \theta_2,\ E(X_2) = 2\theta_1 + \theta_2,\ E(X_3) = \theta_1 + 2\theta_2$$

obtain the unbiased estimates of θ_1 and θ_2. Also obtain their variances and the covariances. 10

(b) In a genetical experiment, the observed frequencies in four possible classes (*AB*, *Ab*, *aB*, *ab*) are reported as 125, 18, 20 and 34. The corresponding probabilities of each group are

$$\frac{1}{4}(3 - 2\theta + \theta^2),\ \frac{1}{4}(2\theta - \theta^2),\ \frac{1}{4}(2\theta - \theta^2) \text{ and } \frac{1}{4}(1 - 2\theta + \theta^2)$$

respectively. Obtain the MLE of θ and its variance. 10

(c) Let

$$X \sim BD(x, p) \text{ and } L(p, d(x)) = (p - d(x))^2$$

If p is assumed to have a uniform prior distribution $(0 < p < 1)$, then obtain the Bayes estimate of p and the corresponding Bayes risk. 10

(d) Let T_1 be unbiased for $g(\theta)$ with $E(T_1^2) < \infty$ and sufficient. Let $h(T) = E(T_1 \mid T)$, then show that—

(i) $E[h(T)] = g(\theta)$

(ii) $V[h(T)] \leq V(T_1)$

equality holding iff $P\{[T_1 - h(T)] = 0\} = 1$. 10

4. (a) Show that the maximum likelihood estimate is a BAN estimator. 10

(b) Let $(X_1, X_2, \cdots, X_n)$ be a random sample from $N(\mu, \sigma^2)$, when both (μ, σ^2) are unknown. Obtain $100(1-\alpha)\%$ confidence interval for σ^2. 10

(c) If $X_1, X_2, \cdots, X_n$ are i.i.d. Poisson (λ), obtain a UMVUE of $g(\lambda) = e^{-\lambda}(1+\lambda)$. 10

(d) Explain shortly Bootstrap and Jackknife methods. 10

Section—B

5. *(a)* An urn contains 10 marbles out of which M are white and $10 - M$ are black. To test $H_0 : (M = 5)$ against $H_1 : (M = 6)$, one draws three marbles from the urn without replacement. The hypothesis is rejected if the sample contains 2 or 3 white marbles otherwise it is accepted. Find the values of α and β. 8

(b) In usual notion, prove that

$$E\{e^{Z_n t}[\phi(t)]^{-n}\} = 1$$

for any point t in the set D. 8

(c) Let X be $N_p(\mu, \Sigma)$. Obtain the characteristic function and show that $Y(= CX)$ is also normal N_p. 8

(d) For a single sampling plan for attributes, obtain the probability of acceptance, $(L(p))$ of the lot for known values of C. 8

(e) If

$$D_n = \sup_x |F_n(x) - F_X(x)|$$

show that it is distribution-free, where $F_n(x)$ is the empirical distribution function and $F_X(x)$ is the cumulative distribution function. 8

6. *(a)* Develop a test for $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ based on a random sample of size n from an exponential distribution with mean $\frac{1}{\theta}$. 10

(b) Discuss the problem of symmetry associated with the testing of several multivariate population mean vectors. 10

(c) Develop a sequential test for $H_0 : p = p_0$ against $H_1 : p = p_1$, if the sampling is done from $B(1, p)$. Also obtain the OC function and ASN function. 10

(d) Develop the discriminant function to discriminate between two normal populations $N(\underset{\sim}{\mu}_{(1)}, \Sigma)$ and $N(\underset{\sim}{\mu}_{(2)}, \Sigma)$. Determine the probabilities of mis-classifications. 10

7. *(a)* In a decision problem

$$\Omega = \left\{ \frac{1}{5}, \frac{1}{2} \right\}, \; A = \{a_1, a_2\}, \; P_\theta(X = 1) = \theta, \; P_\theta(X = 0) = 1 - \theta$$

and the loss table is given as

Ω \ A	a_1	a_2
θ_1	0	10
θ_2	10	0

(i) Find all possible decision rules.

(ii) Find risk corresponding to all decision rules.

(iii) Find minimax decision rules.

(iv) Find a set of admissible decision rules. 10

(b) Define an empirical distribution function. For a random variable with empirical distribution function $F_n(x)$, show that

$$P\left[F_n(x) = \frac{j}{n}\right] = \binom{n}{j}[F_X(x)]^j[1 - F_X(x)]^{n-j},$$

$$j = 0, 1, 2, \cdots, n$$ 10

(c) Define the following terms : 10

(i) AOQ

(ii) AOQL

(iii) LTPD

(iv) ASN and OC function

(v) ATI

(d) Develop a test for independence of sets of variates in a p-variate normal distribution. 10

8. (a) Consider a q-normal population

$$N(\mu^{(i)}, \Sigma_i),\ i = 1, 2, \cdots, q$$

Develop a test for

$$H_0 : \Sigma_1 = \Sigma_2 = \cdots = \Sigma_q = \Sigma$$ 10

(b) Define a single-parameter exponential family of distributions. State the tests for—

(i) $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$

(ii) $H_0 : \theta_1 \le \theta \le \theta_2$ versus $H_1 : \theta < \theta_1$ or $\theta > \theta_2$

Are these tests UMP and unbiased? 10

(c) A coin is tossed 8 times and the total number of heads were observed. A test procedure rejects the null hypothesis that the coin is unbiased. If the number of heads is less than 2 or more than 6, obtain α and power of the test for $p = 0 \cdot 2$. 10

(d) Explain a two-person zero-sum game. State the fundamental theorem of game theory. 10

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